1. The joint density of the two samples is
$$f(\mathbf{x},\mathbf{y};\mu_1,\mu_2) = \mu_1^{-n}\exp(-\sum x_i/\mu_1)\,\mu_2^{-n}\exp(-\sum y_i/\mu_2).$$
The (unconstrained) MLEs are $\hat{\mu}_1 = \bar{X}$ and $\hat{\mu}_2 = \bar{Y}$. Under $H_0 : \mu_1 = \mu_2 = \mu$, the constrained MLE is $\hat{\mu} = (\bar{X}+\bar{Y})/2$. Hence
$$MLR = \frac{f(\mathbf{X},\mathbf{Y};\hat{\mu_1},\hat{\mu_2})}{f(\mathbf{X},\mathbf{Y};\hat{\mu},\hat{\mu})} = \frac{(\bar{X}+\bar{Y})^{2n}/2^{2n}}{\bar{X}^n\bar{Y}^n} = 2^{-2n}\{\sqrt{\bar{X}/\bar{Y}} + \sqrt{\bar{X}/\bar{Y}}\}^{2n}.$$
Under H_0,
$$2\ln MLR = 4n\ln(\sqrt{\bar{X}/\bar{Y}} + \sqrt{\bar{X}/\bar{Y}}) - 4n\ln 2 \sim \chi^2_1$$
approximately. We reject H_0 if $2\ln MLR > \chi^2_{1,\alpha}$, where $\chi^2_{1,\alpha}$ denotes the upper $100\alpha\%$ quantile of the χ^2_1 distribution.

2. For $i = 1,2,3$ and 4, $X_i \sim Bin(p_i, 200)$. We need to test
$$H_0 : p_1 = p_2 = p_3 = p_4 \quad \text{against} \quad H_1 : p_i's \text{ are not all the same.}$$
The likelihood function is
$$L(p_1,p_2,p_3,p_4) = \prod_{i=1}^{4}\frac{200!}{x_i!(200-x_i)!}p_i^{x_i}(1-p_i)^{200-x_i}.$$
Unconstrained MLEs are $\hat{p}_i = x_i/200$, $1 \le i \le 4$. Under H_0, the constrained MLE for common p is $\hat{p} = \sum_{i=1}^4 x_i/(4\times 200)$. Hence
$$MLR = \frac{L(\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4)}{L(\hat{p},\hat{p},\hat{p},\hat{p})} = \frac{\prod_{i=1}^4 \hat{p}_i^{x_i}(1-\hat{p}_i)^{200-x_i}}{\hat{p}^{\sum_i x_i}(1-\hat{p})^{800-\sum_i x_i}}.$$
$$2\ln MLR = \sum_{i=1}^{4}\{x_i\ln\hat{p}_i + (200-x_i)\ln(1-\hat{p}_i)\} - \sum_i x_i\,\ln\hat{p} + (800-\sum_i x_i)\ln(1-\hat{p}),$$
which is approximately χ^2_3 under H_0. (To see the degree of freedom, let $p_i = p_1 + \delta_i$ for $i = 2,3,4$. Then the null hypothesis is $H_0 : \delta_1 = \delta_2 = \delta_3 = 0$.) We reject H_0 if $2\ln MLR > \chi^2_{3,\alpha}$, where $\chi^2_{3,\alpha}$ is the upper $100\alpha\%$ quantile of χ^2_3 distribution.

For the given data, $2\ln MLR = 153.96 > \chi^2_{3,0.05} = 7.815$. Hence we reject H_0 and conclude that there exists significant difference among the four areas.

3. Note $\sum_{i=1}^{10}(X_i-\bar{X})^2/\sigma^2 \sim \chi^2_9$. Leaving out 5% at both ends of the distribution, we have
$$\begin{aligned} 0.9 &= P\{3.325 < \sum_{i=1}^{10}(X_i-\bar{X})^2/\sigma^2 < 16.92\} \\ &= P\{\sum_{i=1}^{10}(X_i-\bar{X})^2/16.92 < \sigma^2 < \sum_{i=1}^{10}(X_i-\bar{X})^2/3.325\}, \end{aligned}$$
namely, $a_1 = 16.92$ and $a_2 = 3.325$.

Similarly since $\sum_{i=1}^{10}(X_i-\mu)^2/\sigma^2 \sim \chi^2_{10}$, we have
$$\begin{aligned} 0.9 &= P\{3.94 < \sum_{i=1}^{10}(X_i-\bar{X})^2/\sigma^2 < 18.31\} \\ &= P\{\sum_{i=1}^{10}(X_i-\mu)^2/18.31 < \sigma^2 < \sum_{i=1}^{10}(X_i-\mu)^2/3.94\}, \end{aligned}$$

namely, $b_1 = 18.31$ and $b_2 = 3.94$.

The average lengths of the confidence intervals are

$$(a_2^{-1} - a_1^{-1})E\{\sum_{i=1}^{10}(X_i - \bar{X})^2\} = (a_2^{-1} - a_1^{-1}) \times 9\sigma^2 = 2.18\sigma^2,$$

$$(b_2^{-1} - b_1^{-1})E\{\sum_{i=1}^{10}(X_i - \mu)^2\} = (b_2^{-1} - b_1^{-1}) \times 10\sigma^2 = 1.99\sigma^2.$$

The second interval is shorter since it makes use of given mean μ.

4. The MLE for θ is $X_{(n)}$, the sample maximum. It is easy to see that for $x \in [0, 1]$,

$$P\{X_{(n)}/\theta < x\} = P\{X_i/\theta < x \text{ for all } 1 \le i \le n\} = \big[P\{X_1/\theta < x\}\big]^n = x^n.$$

Hence $X_{(n)}/\theta$ is a pivot with probability $f(x) = nx^{n-1}$ for $0 \le x \le 1$. To find a $100(1-\alpha)\%$ confidence interval for θ, we need to find a and b such that

$$P\{a \le X_{(n)}/\theta \le b\} = 1 - \alpha.$$

Obviously there are many choices for a and b here. However we prefer the interval which has the shortest length. Therefore we look for an interval on which the probability density $f(x)$ is as large as possible. Hence we should let $b = 1$ and choose a according to α. This yields $a = \alpha^{1/n}$. The resulting confidence interval for θ is

$$[X_{(n)}/b,\ X_{(n)}/a] = [X_{(n)},\ X_{(n)}\alpha^{-1/n}].$$